Duality at Classical Electrodynamics and its Interpretation through Machine Learning Algorithms

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***Abstract*—The aim of this paper is to investigate the hypothetical duality of classical electrodynamics and quantum mechanics through the usage of Machine Learning principles. Thus, the Mitchell's criteria are used. Essentially this paper is focused on the radiated energy by a free electron inside an intense laser. The usage of mathematical strategies might be correct to some extent so that one expects that classical equation would contain a dual meaning. The concrete case of Compton scattering is analyzed. While at some quantum field theories might not be scrutinized by computer algorithms, contrary to this Quantum Electrodynamics would constitute a robust example.**

Keywords*—Classical electrodynamics; quantum mechanics; machine learning principles; mitchell's criteria

I. Introduction

The science of Machine Learning is been applying to a wide spectrum of disciplines in both basic sciences as well as engineering. Mainly the purpose of this application is the improvement of the functionalities of systems. This is often linked to a kind of optimization of the critic variables of systems. Thus to get the best scenario for each system one needs firstly to identify the relevant pieces that would play a critic role in the chain of processes. Clearly one here would argue that Machine Learning is actually applicable to all those input-output systems whose black-box might be unknown. In this manner emerges the necessity of differentiating the one-way path that do not allows to come back at the beginning of the processes. Among the plethora of Machine Learning philosophies one finds the one invented by Tom Mitchell that establishes that system can learn from a triplet of postulates:

- All system has explicitly a concrete task that allow it to develop in a sustained manner.
- In order to accomplish the nominal task the system must to apply a coherent strategy based on a methodology that would have to exhibit a well-designed performance.
- Once the system has accomplished its task then it would analyze if the performance of the used strategies were the right ones against other alternatives. Only if the task was solved without to expends the system resources then one calculated the efficiency of the involved processes. When this is high enough the one can say that the system has enough learning to be applied successively.

Motivated by the Mitchell's criteria, in this paper the energy radiated by a free electron inside an external super intense laser is treated with these criteria. In essence the study is centered in the following question: Given a relativistic electron in a strong laser, under what conditions the classical physics is abandoned to pass a entire scenario governed by quantum mechanics.

These so-called hybrid theories that combines criteria from classical physics and quantum mechanics have been studied at an entire framework of quantum electrodynamics (QED in short) by R. P. Feynman [1]. Subsequently have appeared the works of Volkov [2], Narozhnyi [3], Vachaspati [4], Kibble [5], Reiss [6], Eberly [7] that have studied QED with infinite waves that can be seen as classical fields without quantization. For example consider the QED Lagrangian:

$$\mathcal{L} = -ie \int dx^4 \bar{\Psi} \gamma_\mu A^\mu \Psi \tag{1}$$

where the spinors $\bar{\Psi}$ and Ψ satisfy the Dirac equation and A^μ the external field expressed as an 4-dimension vector without any type of quantization in contrast with the Dirac spinors. As it is well-known in field theories, from Eq.1 one can extend it to others types of elementary interactions by which is usual to derive the well-known diagrams of Feynman.

This paper is entirely focused on the implications of the vectorial potential $\mathbf{A}$ inside classical electrodynamics that allows to estimate observables such as energy radiated as well as to make predictions at the generation of new sources of powerful light at the super-intense regime. Thus emerges the following questions:

- Under what conditions the energy radiated by an electron is quantized?
- There is an exact boundary that separates the quantum mechanics and classical description of radiation emitted by an relativistic electron?
- Is the interaction electron and external field a system that can be described by the principles of Machine Learning?

This paper explores the capabilities of Machine Learning [8] to measure the limits between classical electrodynamics and QED in the concrete case when a relativistic electron is inside a external super-intense laser. For this end the Mitchell criteria [9] are employed to distinguish the scenarios where a transition from the classical to quantum takes place. In second section the theoretical machinery is presented. In third section the implementation of Mitchell criteria at the classical formalism and its link to QED is presented and discussed. In last section the conclusion of paper is presented.

II. Theoretical Machinery

A. Classical Backscattering Radiation

An important example of the transition from classical to quantum dynamics of light is given by the classical Compton backscattering. Commonly the theory demands to employ the covariant notation given by ϕ=$k^{\mu}x_{\mu}$=$k\cdot x$ based from the definition x_{μ}=(x_0,$\vec{x}$). From this the 4-vector k^{μ}=(1,0,0,1) describes an incident field along the +z direction yielding ϕ=x_0-z. Thus the 4-vector $A_{\mu}\equiv$ (0,$\vec{\mathbf{A}}$) with $\vec{\mathbf{A}}$= $\mathbf{A}_x(\phi)$. For the concrete case of backscattered radiation the direction must be opposite to the incident field and written as -$\hat{k}$, which is means that the emitted radiation travels along the -z direction. With the definition of $\xi=\frac{e^2u_0^2\chi^2}{4\pi^2}$ one can write down:

$$\frac{d^2I(\omega,\mathbf{n}\Rightarrow -z)}{d\omega d\Omega}=\xi\left|\int_{-\infty}^{+\infty}A_x(\phi)\exp\left\{i\chi\left[\phi+\int_{-\infty}^{\phi'}\mathbf{A}^2(\psi)d\psi\right]d\phi\right\}\right|^2. \quad (2)$$

That is the backscattered radiation intensity derived in [10] with χ the shifted Doppler frequency that can be interpreted in classical electrodynamics as the harmonics of radiated energy and at the quantum mechanics language as the number of emitted laser photons. One can see that the quantity $|...|^2$ contains all the information of the processes of classical radiation. Although above nonlinear Compton scattering was derived from Eq. 2, then one can speculate about the possible quantum mechanics that it might contain. Now one can go through the integration of the exponential which is the focus of this paper. In the case of linear polarization the laser field which is assumed to be super-intense is defined as $\mathbf{A}(\psi)$=$a\sin(\psi)\vec{i}$ then the integrand is written as $a^2\int_{-\infty}^{\phi'}\sin^2(\psi)d\psi$. The integration can be done in a straightforward manner yielding the product of three exponential:

$$\exp\left\{i\chi\left[\phi+\int_{-\infty}^{\phi'}\mathbf{A}^2(\psi)d\psi\right]d\phi\right\}$$
$$=\exp(i\chi\phi)\exp(i\frac{i\chi a^2}{2}\phi')\exp(-i\frac{\chi a^2}{2}\sin2\phi') \quad (3)$$

These changes have as objective to create infinite series using the basis of integer-order Bessel's functions guided by the formulation of Ritus and Nishikov [11] as follows:

$$\exp(i\chi\sin\phi)=\sum_{\ell}J_{\ell}(\chi)\exp(i\ell\phi) \quad (4)$$

$$\exp(i\frac{\chi a^2}{2}\sin\phi')=\sum_{m}J_m(\frac{\chi a^2}{2})\exp(im\phi') \quad (5)$$

$$\exp(-i\frac{\chi a^2}{2}\sin2\phi')=\sum_{n}J_n(\frac{\chi a^2}{2})\exp(-in2\phi') \quad (6)$$

the usage of this technique that favorably ends in a kind of quantization of the intense field but working in a fully QED scenario given by the Volkov's states. Thus, with all these expansions and inserting them in Eq.3 and therefore inserting the result in Eq. 2 then one arrives to an important relation written below as:

$$\frac{d^2I(\omega,-z)}{d\omega d\Omega}=\xi\left|\int_{-\infty}^{+\infty}A_x(\phi)\sum_{\ell}J_{\ell}(\chi)\exp(i\ell\phi)\sum_{m}J_m(\frac{\chi a^2}{2})\exp(im\phi')\sum_{n}J_n(\frac{\chi a^2}{2})\exp(-in2\phi')d\phi\right|^2$$
$$=\xi\left|\int_{-\infty}^{+\infty}A_x(\phi)\sum_{\ell}\sum_{m}\sum_{n}J_{\ell}(\chi)J_m(\frac{\chi a^2}{2})J_n(\frac{\chi a^2}{2})\exp(i\ell\phi)\exp(im\phi')\exp(-in2\phi')d\phi\right|^2. \quad (7)$$

It should be noted that Eq. 7 is a fully classical relation so that in a first instance it is impossible to link it to any fact done inside the quantum mechanics framework. In fact, one can see the it is pure emitted radiation done by a relativistic electron. Nevertheless the work of Ritus [12] it was proposed the connection between a semi-classical description a quantization of external light. In order to follow the Ritus's view the covariant quantities are explicitly written as $\phi'=k'_{\mu}x^{\mu}$ and $\phi=k_{\mu}x^{\mu}$. With this the argument of product of exponentials in Eq. 7 is written as: $i\ell k_{\mu}x^{\mu}+imk'_{\mu}x^{\mu}-2ink'_{\mu}x^{\mu}$. When it is conveniently ordered then one gets $i[\ell k_{\mu}-(2n-m)k'_{\mu}]x^{\mu}$. Clearly one has only information of light either emitted or absorbed despite the fact that the electron is the responsible of these processes. To homogenize the physics of this event it is also convenient to introduce the exponential $\exp[i(p_{\mu}-p'_{\mu})x^{\mu}]$. To note that it is possible only if $p_{\mu}-p'_{\mu}\approx 0$ at the space-point x^{μ}. Logically the purpose of this is twofold:

- The conservation of 4-momentum.
- To force a kind of quantization of external field.

In this manner the argument of the resulting product can be written below as:

$$\text{Exp}\left[i(p_{\mu}+\ell k_{\mu}-(2n-m)k'_{\mu}-p'_{\mu})\,x^{\mu}\right. \quad (8)$$

Thus one can easily to recognize that there is a pure conservation with the initial and final states of 4-momentum given by:

$$\mathbf{P}_{\text{IN}}=p_{\mu}+\ell k_{\mu}, \quad (9)$$
$$\mathbf{P}_{\text{FI}}=(2n-m)k'_{\mu}+p'_{\mu}. \quad (10)$$

As mentioned above the fact that emerge integer numbers it might not be directly linked to a case of quantization as noted by Ritus. Thus this kind of artificial quantization in a theoretical framework would have to be verified experimentally within a valid window of accuracy [13].

III. The Machine Learning Analysis

Although classical electrodynamics has been entirely developed as a robust branch of physics, the requirement of using advanced algorithms to minimize a biased interpretation of the equations might be seen as an advantage more than a disadvantage at the sense that one gets a kind of hybrid theory with a tuned interpretation. In the following the well-known Mitchell's criteria shall be used to provide a fair interpretation of Eq. 7. These criteria are classified and conceptualized as follows:

- The Task: any system might to have one or more tasks that justifies its existence.
- The Performance: once the task is identified, the system opts by a strategy that must to exhibit a well-defined performance.
- The Experience: depending on the performance and the completion of task, the system should be able to assess the experience along the chain of events. At the cases of an acceptable experience the system can claim that it was a kind of learning to be repeated in subsequent task.

Turning back to Eq. 7, then one can wonder if it requires to isolate a concrete task? And if it is required then for what? One can argue that the manner as Eq. 7 is written, then it represents already a **task** in the sense that one should figure out the best way to extract a reasonable interpretation through a rather clear procedure of integration without any ambiguity. With Eq. 9 and Eq. 10, Eq. 7 can now be written as:

$$\frac{d^2I(\omega,-z)}{d\omega d\Omega}=\xi\left|\int_{-\infty}^{+\infty}A_x(\phi)\sum_{\ell,m,n}J_\ell(\chi)J_m(\frac{\chi a^2}{2})J_n(\frac{\chi a^2}{2})\exp[-i(p_{\rm IN}-p_{\rm FI})_\mu x^\mu]d\phi\right|^2. \tag{11}$$

In addition, the task can also be seen as the demonstration that Eq. 7 is a hybrid formulation of radiated energy by a relativistic free electron. In other words, one can also wonder about the concrete capabilities of classical electrodynamics to exhibit a dual formulation of a fundamental process such as Compton scattering (or it can also be Thomson scattering [14] to some extent, for instance). From Eq. 11 one can wonder if it is quantum mechanics expression or it is needed to corroborate such hypothesis. By following the Mitchell's criteria it is desirable to define a clear route to argument that in fact Eq. 11 is a hybrid expression so that quantum mechanics laws apply. From Eq. 11 to demonstrate that it is a quantum mechanics description of process of interaction between a field and a relativistic free electron, the proposed **performance** requires only the verification at the energy or momentum. For the sake of simplicity the present analysis shall use the energy integration. This demands to employ $\phi=k^\mu x_\mu=\omega t$ that gives $d\phi$=$td\omega$ under the assumption that the photon 4-vector momentum is (1,0,0,0). Thus one arrives to:

$$i[\mathcal{E}+\ell\omega-(2n-m)\omega'-\mathcal{E}']t. \tag{12}$$

With Eq. 12 one clearly finds that its inclusion in Eq.11 turns out to be a Dirac-delta function in the sense that one gets:

$$\int\exp[i(\ell\omega+\mathcal{E}-(2n-m)\omega'-\mathcal{E}')t]\omega dt$$
$$\omega\delta(\ell\omega+\mathcal{E}-(2n-m)\omega'-\mathcal{E}'). \tag{13}$$

Eq. 13 is an important result in the sense that at least at the energy variable one finds conservation if only if the argument of Dirac-delta function is null. By inserting Eq. 13 into Eq. 11 one arrives to:

$$\frac{d^2I(\omega,-z)}{d\omega d\Omega}=\xi\omega^2a^2\left|\sum_{\ell,m,n}J_\ell(\chi)J_m(\frac{\chi a^2}{2})J_n(\frac{\chi a^2}{2})\delta[(\ell+1)\omega+\mathcal{E}-(2n-m)\omega'-\mathcal{E}']\right|^2. \tag{14}$$

Eq. 14 can be normalized to introduce into a scenario of probabilities. In this manner it is needed that:

$$N^2\frac{d^2I(\omega,-z)}{d\omega d\Omega}=1. \tag{15}$$

Therefore, the constant N can be explicitly written as:

$$N=\sqrt{\frac{1}{\xi\omega^2a^2\left|\sum_{\ell,m,n}J_\ell(\chi)J_m(\frac{\chi a^2}{2})J_n(\frac{\chi a^2}{2})\delta[(\ell+1)\omega+\mathcal{E}-(2n-m)\omega'-\mathcal{E}']\right|^2}}. \tag{16}$$

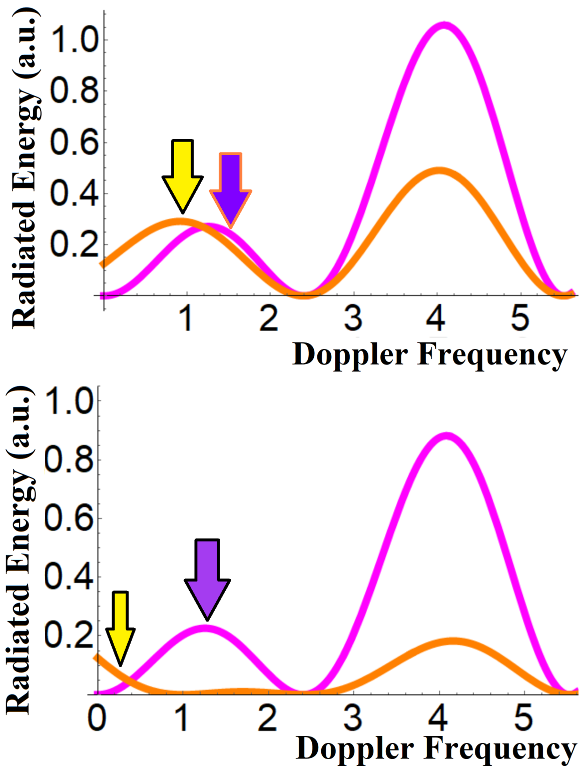

Fig. 1. Classical Distributions of Radiated Energy: Up: $|H(\chi,a)|^2$=0.16$|J_0(\chi)J_1$ (0.001χ) $J_0(0.001\chi)|^2$ (Magenta Color), and $|H(\chi,a)|^2$=0.5$|J_0(\chi+0.01)J_1$ $(0.01\chi+0.01)$ $J_0(0.01\chi+0.01)|^2$ (Orange Color). Down: $|H(\chi,a)|^2$ = 0.52×10^7 $J_0(\chi)$ × $J_1(0.001\chi)$ × $J_0(0.001\chi)$ (Orange Color) and $|H(\chi,a)|^2$ = 0.5×10^4 $J_0(\chi-0.01)$ × $J_1(0.01\chi-0.01)$ × $J_0(0.01\chi-0.01)$ (Magenta Color).

Once the constant N is calculated then the radiated energy can be estimated fom a procedure based entirely at the Mitchell criteria. Therefore, one can pass the classical electrodynamics concepts to one dictated by principles of Machine Learning.

IV. Interpretation by Mitchell's Criteria

In this way one gets that the applied **performance** has brought elements that play a concrete role at quantum electrodynamics (or QED in short). Thus Eq. 14 makes us to remind the well-known diagrams of Feynman's. In effect, the concordance with QED [15] as seen at the argument of Dirac-delta function has as central implication this pseudo quantization of electromagnetic field that although proposed initially as classically pure, now the interpretation of Feynman's rules [16] would suggest that in the initial state a free electron with initial energy $\mathcal{E}$ absorbs $[\ell+1]$ photons, whereas at the corresponding final state the electron has an energy $\mathcal{E}'$ and has emitted $[2n-m]$ photons. Actually, it is imminently nonlinear Compton scattering as observed by Bula [17]. Since Eq. 14 has been recognized as a potential expression that would play a role in QED, then one expects to arrive to a solid experience after the assumptions that the performance has demanded. In this way one can propose that Eq. 14 is a kind of square of sum of all allowed amplitudes that certainly it could to involve both linear as well as nonlinear contributions [18][19][20][21][22][23][24]. The case of simple Compton is of particular interest. Eq. 14 can be Compton scattering is $\ell=0$ and $2n-m=1$. However one can see that the crude assumption that these integer number would denote the number of photons fails because it is required that $n=m/2$ fact that is totally false in quantum mechanics. This reveals the "**bugs**" of algorithm to propose an effective strategy or performance. Thus simple Compton is restored with $\ell=n=0$ and $m=1$. When $H(\chi,a)$=$J_0(\chi)J_1(\frac{\chi a^2}{2})J_0(\frac{\chi a^2}{2})$ the classical radiated energy can be also interpreted as the measured quantum mechanics observable $\mathcal{O}(\chi,a)$, while $|H(\chi,a)|^2$ the square of all possible amplitudes. Therefore one can write down (where it is assumed after the integration of Dirac-delta function as commonly done in QED):

$$\frac{1}{\xi\omega^2a^2}\frac{d^2I(\omega,-z)}{d\omega d\Omega}=\mathcal{O}(\chi,a)$$
$$=\left|H_{m=1}(\chi,a)\delta[\omega+\mathcal{E}-\omega'-\mathcal{E}']\right|^2\approx|H_{m=1}(\chi,a)|^2 \quad (17)$$

A. Generation of Pseudo Amplitudes

In Fig. 1 (Up) one can see the plotting of $|H(1,0,0,\chi,a)|^2$=$|J_0(\chi)J_1(\frac{\chi a^2}{2})J_0(\frac{\chi a^2}{2})|^2$ for two cases: (i) the magenta-color line denoting simple Compton scattering with $\xi\omega^2a^2$=0.1610^7 and $a^2=0.002$ expressing the fact that the laser is not super-intense as initially assumed. (ii) the orange-color line is given by: $|H(1,0,0,\chi,a)|^2$=$|J_0(\chi+0.01)J_1(\frac{\chi a^2}{2}+0.01)J_0(\frac{\chi a^2}{2}+0.01)|^2$ resulting that the peak is shifted to the left-side with a rough coincidence to the value of argument of Bessel function. The added value 0.01 can be perceived as the error at the measurement of Doppler frequency. The reader can corroborate that the case of using -0.01 the negative case, the first peak or Compton peak is gone. In Fig. 1 (Down) the case of $|H(\chi,a)|^2$ = 0.52×10^7 $J_0(\chi)$ × $J_1(0.001\chi)$ × $J_0(0.001\chi)$ (orange color) and $|H(\chi,a)|^2$ = 0.5×10^4 $J_0(\chi-0.01)$ × $J_1(0.01\chi-0.01)$ × $J_0(0.01\chi-0.01)$ (magenta color) is plotted. While the magenta color distribution exhibits the fact that the central peak is shifted to right-side, the orange color distribution the incorporates errors at the order of 0.01 at the arguments of Bessel function given by: $0.01\chi-0.01$ one can perceive this as the degradation of radiation spectra due to quantum mechanics effects. The product of Bessel functions from above can be defined as a kind of pseudo amplitudes that can be written as:

$$H(m,n,\ell,\chi,a)=\left|J_\ell(\chi)J_m(\frac{\chi a^2}{2})J_n(\frac{\chi a^2}{2})\right|. \quad (18)$$

where $J_\ell(\chi)$ acts as a propagator whereas $J_m(\frac{\chi a^2}{2})$ and $J_n(\frac{\chi a^2}{2})$ can be understood as a kind of input and output states. In fact, the integer number given by the order of Bessel function, is expressing the fact that there is a kind of "classical" absorption as well as emission. From Eq. 17, the square $|H(m,n,\ell,\chi,a)|^2$ represents an observable that is related to radiated energy. This is of importance at the sense that Machine Learning can manage the best values of integer number in order to find the peaks of radiation that to some extent can be perceived as the peaks of X-rays [25][26][27].

In this manner one has below:

$$H(\chi, a) = \sum_{m,n,\ell} \left| J_\ell(\chi) J_m(\frac{\chi a^2}{2}) J_n(\frac{\chi a^2}{2}) \right|, \qquad (19)$$

that emulates to some extent the sum of all possible possibilities for absorption and emission. Then, the intensity of radiated energy is the square of Eq. 19:

$$I(\chi, a) = \left| \sum_{m,n,\ell} J_\ell(\chi) J_m(\frac{\chi a^2}{2}) J_n(\frac{\chi a^2}{2}) \right|^2. \qquad (20)$$

From Eq. 20 one can define the purity of emission at the sense that a detector can sense Compton photons accompanied of pile-up photons, all those that were created by parallel processes. Then this purity is written as:

$$P(\chi, a) = \frac{\left| \sum_{m,n,\ell} J_\ell(\chi) J_m(\frac{\chi a^2}{2}) J_n(\frac{\chi a^2}{2}) \right|^2}{\left| \sum_{m,n,\ell} J_\ell(\chi) J_m(\frac{\chi a^2}{2}) J_n(\frac{\chi a^2}{2}) \right|^2 + \frac{d^2 I_{\rm B}(\omega_{\rm B}, a_{\rm B})}{d\omega_{\rm B} d\Omega}} \qquad (21)$$

where $I_{\rm B}$, $\omega_{\rm B}$ and $a_{\rm B}$, the intensity, frequency and intensity of background field. In praxis one expects actually that the Compton photons have greater energy than their noise in order to be efficiently detected (see for example [28][29][30][31][32][33]). Thus one has below that:

$$\frac{\frac{d^2 I_{\rm B}(\omega_{\rm B}, a_{\rm B})}{d\omega_{\rm B} d\Omega}}{\left| \sum_{m,n,\ell}^{M,N,L} J_\ell(\chi) J_m(\frac{\chi a^2}{2}) J_n(\frac{\chi a^2}{2}) \right|^2} << 1. \qquad (22)$$

In this manner, while the Machine Learning **task** would consist in the identification of Compton photons still at a classical scenario. Then, the **performance** would need to have a clear procedure to accomplish the identification of that photons (see for example [34][35][36][37][38]). Thus, it would consist in the searching of the best values of M, N, L to satisfy Eq. 22. Subsequently, the **experience** would be that of Eq. 21. The final Sigmoid function can be derived from Eq. 21 and Eq. 22 so that one arrives to:

$$S(\chi, a, M, N, L) = \frac{1}{1 + {\rm Exp}\left[-\frac{\frac{d^2 I_{\rm B}(\omega_{\rm B}, a_{\rm B})}{d\omega_{\rm B} d\Omega}}{H(\chi, a)} \right]}. \qquad (23)$$

The sign "-" in Eq. 23 emerges from the fact that $H(\chi, a)$ can acquire negative values due to the Bessel functions. A deeply analysis of Eq. 23 can be a window to investigate all those available values of classical radiation that might be encompassing quantum mechanics. In addition, the role of these integer number can also be correlated to the existence of a kind of entropy that would appear from the projection of classical observables in a quantum scenario, in the sense that Entropy = Log$[|H(\chi, a)|^{g(r)}]$ (see for example [39], Fig. 2).

V. Conclusion

In this paper, the derivation of quantum observables have been possible with the classical electrodynamics of Hartemann-Kerman equation. The radiated energy equation has been derived, and its relevance in quantum mechanics has been done through the criteria of Tom Mitchell. Along rhis document, integer number have been obtained in a fully analogy to the states of absorption and emission of photons of a relativistic electron in a laser field. Thus, the resulting spectra are strongly dependent on the integer-order Bessel functions. Therefore, the integer number are exploited at the sense that them allow to design a strategy inside the territory of Machine Learning. Finally a Sigmoid function was derived. This clearly demonstrates that classical electrodynamics appears to exhibit a certain flexibility to be adapted to new concepts of computing in physics theoretical as well as experimental.

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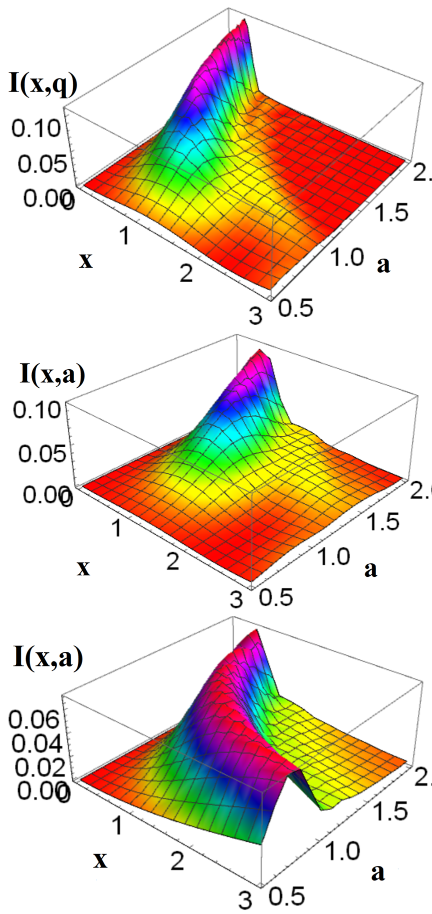

Fig. 2. Illustration of Intensity $I(\chi, a)$=$H(m, n, \ell)$ for $M = N = L$ =2 (Up), $M = N = L = 10$ (Middle) and M, N, L=15 (Down). It is Noteworthy the Formation of a Central Peak Starting from the Lowest Values of Both a as well as the Normalized Doppler-Shifted Frequency.

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