Computational Study of Quantum Coherence from Classical Nonlinear Compton Scattering with Strong Fields

Huber Nieto-Chaupis
Universidad Autónoma del Perú
Panamericana Sur Km. 16.3 Villa el Salvador
Lima Perú

***Abstract*—From the covariant formulation of radiation intensity of Hartemann-Kerman model entirely constructed in the classical electrodynamics scenario, a formulation of coherent states has been obtained in an explicit manner represented by the infinite sum of integer-order Bessel functions. Both linear and nonlinear Compton scattering are included, suggesting that Compton processes can be perceived as coherent states of light-matter interaction.**

***Keywords*—Quantum coherence; bessel; compton scattering**

I. Introduction

Compton scattering is seen as a "golden processes" inside Quantum Electrodynamics [1]. This has played in the understanding role in the understanding of quantum mechanics of light-matter interactions. In fact it is pure quantum effect in the which the electron absorbs one single photon and emits one photon with different kinematics than the first one. In a full quantum theory, the Lagrangian of interaction can be written as:

$$\mathcal{L}_{\rm INT} = -ie \int dx^4 \bar{\Psi} \gamma_\mu A^\mu \Psi, \tag{1}$$

with $\bar{\Psi}$ and Ψ the final and initial states, while γ the 4×4 matrices, and A^μ the 4-vector potential that satisfies the Lorentz's gauge $\partial_\mu A^\mu = 0$. In a nutshell the incorporation of a propagator in Eq. (1) and the subsequent operations yields commonly the well-known Feynman's diagrams [2]. Compton scattering was also boarded at the scenario of strong electromagnetic fields. In this case the states $\bar{\Psi}$ and Ψ are solutions of Volkov and obey the equation of Dirac with a external field. In the scenario of high regime where the incoming electromagnetic field has a high density of photons it is usual to define the intensity parameter:

$$\xi^2 = \frac{e^2 \mathbf{A}^2}{m}. \tag{2}$$

introduced by I.I. Goldman [3] who derived the energy of emitted photon given by:

$$\omega' = \frac{2nE\omega}{E(1+\mathrm{Cos}\theta) + [2n\omega + \frac{m^2(1+\xi^2)}{E}(1-\mathrm{Cos}\theta)]}, \tag{3}$$

where the product $n\omega$ is denoting the absorption of n photons. From this $n = 1$ the Klein-Nishina formula is restored. The integer n is linked to the nonlinear processes at the which the electron can absorb various photons simultaneously. These non-linearities can be compacted in the language of Quantum Electrodynamics (QED in short) as a Dirac-Delta function:

$$\delta(E_{\rm I} + n\omega - E_{\rm F} - m\omega'), \tag{4}$$

that indicates the conservation of energy with m an integer number. This non-linearity is theoretically obtained in the emission and absorption of various laser photons by Reiss [4] and Ritus [5] whom have derived quantization of laser in a semi-classical arena with the laser modeled through an circularly polarized infinite wave. This was also seen at [6] and the works of Eberly [7]. A noteworthy attention was paid at the 90s because the prospective construction of a photon-photon collider [8] and the potential apparition of non-linearities as corroborated at the experiments observed at SLAC [9] where nonlinear Compton was observed with strong lasers supporting the fact that these processes can be well modeled by an infinite classical monochromatic wave. In photon collisions one expects that the Compton backscattering can create new particles in according to the reactions:

$$\gamma + \gamma \Rightarrow \tilde{\ell}^+ + \tilde{\ell}^- + \sum_q^Q \Xi_q$$

with the production of Q particles, was predicted inside the framework of new physics of elementary particles [10]. In 1996 nonlinear Compton backscattering have been obtained in an entire arena of classical electrodynamics by Hartemann and Kerman [11] (HK model in short) from the intensity of radiation $\frac{dI(\omega)}{d\Omega} =$

$$\frac{\omega^2}{4\pi^2 c^2} \left| \int dt \int d^3x \mathbf{n} \times [\mathbf{n} \times \mathbf{J}(\mathbf{x}, t)] e^{i\omega[t - \frac{\mathbf{n}\cdot\mathbf{x}}{c}]} \right|^2. \tag{5}$$

Here, it was shown that Compton scattering governs the low intensity whereas in super-strong fields the nonlinear Compton scattering emerges as the apparition of high harmonics that are interpreted as emission of photons with different frequencies. Based at all this background where quantum effects can be retrieved from classical formalisms, this paper tries to derive the quantum coherence from the HK theory. Inspired at the theory of Glauber [12][13], coherent states proportional to Bessel functions are derived. In second section Compton processes are derived from classical electrodynamics. In third section the quantum coherence is derived. Finally at last section the conclusion of paper is presented.

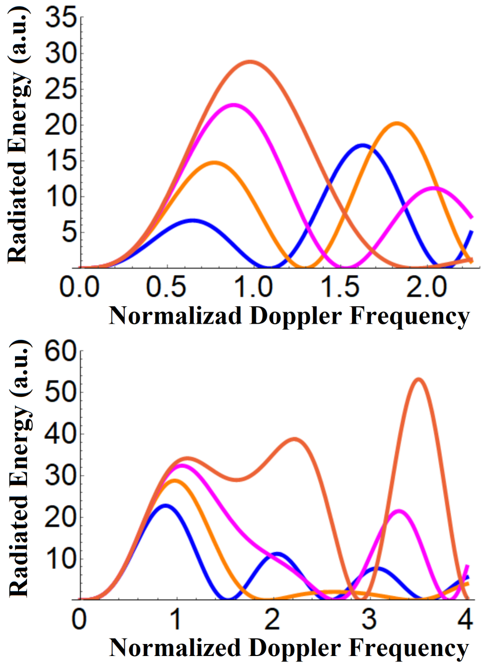

Fig. 1. Classical Distributions of Radiated Energy from Eq. 7 Once the Bessel Expansion as Done at Eq. 14 are Plotted. Up: q Runs from 2 to 5. Down: q from 4 to 7. It should be Noted the Peaks at χ=1.0, Denoting the Simple Classical Compton Scattering.

II. Classical Nonlinear and Linear Compton Scattering

The HK model was developed under a covariant framework in the sense that $\phi = k_\mu \cdot x^\mu$ yielding the well-known radiation intensity of a single electron in an external super intense polarized laser. The radiation intensity is depending on the Doppler-shifted frequency χ. As seen in HK model, Compton and nonlinear Compton scattering was obtained for different values of pulse width. Therefore, one can arrive to the fundamental equation of HK model that can be written down as the distribution of energy radiated per solid angle and frequency and with the definition:

$$\lambda = \frac{e^2}{4\pi^2} u_0^2 \chi^2, \tag{6}$$

then the fundamental equation of covariant HK model can be written as:

$$\frac{d^2 I}{d\omega d\Omega} = \lambda \left| \int_{-\infty}^{+\infty} A_x(\phi) \exp\left\{ i\chi \left[\phi + \int_{-\infty}^{\phi} \mathbf{A}^2(\psi) d\psi \right] d\phi \right\} \right|^2. \tag{7}$$

Clearly one can appeal to different mathematical approaches to extract the quantum mechanics (if any) of Eq. 7 in different ways. At [14], from the HK model the argument of Dirac delta functions have been obtained as well as interpreted as the absorption and emission of photons even when the external field was an infinite wave as commonly expected from QED. Although of course the applicability of advanced mathematical methodologies cannot guarantee not any kind of quantization of external field, a suitable methodology turns out to be the usage of the Fourier expansion. In fact, consider the identity based in the series of Fourier-Bessel so that from the exponential of Eq.7 one gets:

$$\mathrm{Exp}\left\{ i\chi \left[\phi + \int_{-\infty}^{\phi} \mathbf{A}^2(\psi) d\psi \right] \right\} = \sum_{\infty}^{\infty} J_q(\chi) \mathrm{Exp}[iq\theta], \tag{8}$$

with the usage of the crude approximation:

$$\sin\theta = \left[\phi + \int_{-\infty}^{\phi} \mathbf{A}^2(\psi) d\psi \right] \tag{9}$$

$$\Rightarrow \theta = \sin^{-1} \left[\phi + \int_{-\infty}^{\phi} \mathbf{A}^2(\psi) d\psi \right], \tag{10}$$

that to some extent θ can be seen as a phase. In this manner by putting Eq. 8 and Eq. 9 into Eq. 7 one can see that the Bessel functions can be out of the integration. With this Eq. 7 can be written in a more transparent manner as:

$$\frac{d^2 I}{d\omega d\Omega} = \lambda \left| \sum_{-\infty}^{+\infty} J_q(\chi) \right|^2 \left| \int_{-\infty}^{+\infty} d\phi A_x(\phi) \mathrm{Exp}\left(iq\mathbf{sin}^{-1}[\theta(\phi)] \right) \right|^2. \tag{11}$$

As expressed in the HK model, the external field is a super intense laser that is characterized by the width $\Delta\phi$ that is entirely an experimental input. Therefore one can define a function depending on ϕ in the sense that:

$$F(\Delta\phi) = \left| \int_{-\infty}^{+\infty} d\phi A_x(\phi) \mathrm{Exp}\left(iq\mathbf{sin}^{-1}[\theta(\phi)] \right) \right|^2. \tag{12}$$

It is because once the integration is done through the variable ϕ it yields only a pure dependence on the pulse's width $\Delta\phi$ then one can rewrite Eq.11 as:

$$\frac{d^2 I}{d\omega d\Omega} = \lambda \left| \sum_{-\infty}^{+\infty} J_q(\chi) \right|^2 F(\Delta\phi). \tag{13}$$

Subsequently, one can arrive to a normalized backscattered spectrum that would depend on the Doppler-shifted frequency χ. On the other hand by knowing the input value for $\Delta\phi$ then $F(\Delta\phi)$ can opt a finite value, for instance ρ. When λ is written in an explicit manner from Eq. 6 and inserting it into Eq. 13 then the resulting radiation intensity can be written as:

$$\frac{4\pi^2}{e^2 \mu_0^2 \rho} \frac{d^2 I}{d\omega d\Omega} = I(Q, \chi) = \chi^2 \left| \sum_{q}^{Q} J_q(\chi) \right|^2. \tag{14}$$

The way as it is written Eq. 14 allows to displayed it in a straightforward manner. In fact, in Fig. 1 the normalized backscattered spectrum is plotted for two scenarios. Here $\frac{4\pi^2}{e^2\mu_0^2\rho} \approx \xi$. For this exercise, the Up-panel displays various curves of radiated classical energy. Here the sum ran from 2 to the 5th harmonic as given by:

$$I(Q,\chi) = 25\chi^2 \left|\sum_{q=2}^{Q} J_q(3.2\chi+1)\right|^2. \tag{15}$$

One can see there, the Grey color line is denoting the sum of all 4 orders, and it is peaked denoting the fact that still at the classical formulation, scattering Compton can be derived. At the Down-panel where the sum runs from 4 to 7, is exhibiting for instance the Grey liine, a deformed shape in contrast to Up-panel.

$$I(Q,\chi) = 25\chi^2 \left|\sum_{q=4}^{Q} J_q(3.2\chi+1)\right|^2. \tag{16}$$

In on the other side, the Grey line the sum of all four orders, is revealing that high orders might be distorting the peaked centered at χ=1. It is because the highest order are certainly connected to nonlinear Compton scattering. In fact, such deformation at the Grey line is due also to the contribution of more photons to the state of absorption by the electron at the strong electromagnetic field, so that the electron has much energy to emit. Of course, although it is argued in a fully classical scenario, the implementation of integer-order Bessel functions, allows to examine the radiated energy spectra from this perspective. Under the hypothesis that Eq. 14 is an element of an infinite sum then one can generalize it with the change $\xi = \frac{4\pi^2}{e^2\mu_0^2\rho}$ one can write below that:

$$1 + \xi\frac{d^2 I}{d\omega d\Omega} \approx 1 + \frac{\chi^2}{2!}\left|\sum_{-\infty}^{+\infty} J_q(\chi)\right|^2 + ... +$$

$$\frac{\chi^n}{n!}\left|\sum_{-\infty}^{+\infty} J_q(\chi)\right|^n \Rightarrow \xi\frac{d^2 I}{d\omega d\Omega} = \mathrm{Exp}\left\{\left|\chi\sum_{-\infty}^{+\infty} J_q(\chi)\right|^2\right\}, \tag{17}$$

so that Eq. 9 with the hypothesis that $\chi^n = 0$ for $n \geq 3$ thus it can finally be written as:

$$\xi\frac{d^2 I}{d\omega d\Omega} = \mathrm{Exp}\left\{-\left|\chi\sum_{-\infty}^{+\infty} J_q(\chi)\right|^2\right\}. \tag{18}$$

III. Derivation of Quantum Coherence

The mathematical structure of Eq. 10 allows to link it to the quantum mechanics territory in the sense that quantum coherence can be extracted. For this one should assume the following hypothesis:

$$\left|\chi\sum_{q=-\infty}^{Q=+\infty} J_q(\chi)\right|^2 = \frac{\alpha^2}{2}, \tag{19}$$

by which one arrives to:

$$\xi\frac{d^2 I}{d\omega d\Omega} = \mathrm{Exp}\left(-\frac{|\alpha^2|}{2}\right), \tag{20}$$

that in the quantum scenario on gets:

$$|\langle 0|\alpha\rangle|^2 = \mathrm{Exp}\left(-\frac{|\alpha^2|}{2}\right), \tag{21}$$

with α the eigenvalue of the equation $\hat{\mathbf{a}}\,|\alpha\rangle = \alpha\,|\alpha\rangle$. From Eq. 20 and Eq. 21 one arrive to:

$$\xi\frac{d^2 I}{d\omega d\Omega} = |\langle 0|\alpha\rangle|^2 = \mathrm{Exp}\left\{-\left|\chi\sum_{-\infty}^{+\infty} J_q(\chi)\right|^2\right\}, \tag{22}$$

so that one finds that the quantum mechanics amplitude can be written in terms of classical electrodynamics observables:

$$|\langle 0|\alpha\rangle| = \sqrt{\xi\frac{d^2 I}{d\omega d\Omega}}, \tag{23}$$

and the eigen values of coherence can be expressed in terms of interger-order Bessel functions:

$$\alpha^2 = 2\left|\chi\sum_{-\infty}^{+\infty} J_q(\chi)\right|^2, \tag{24}$$

indicating that the values of coherence depend on the χ variable, the normalized Doppler-shifted frequency. It should be noted the relevance of orthogonal polynomial at the classical formulation of coherence [15][16][17][18] In other words, the coherence is directly linked to the frequencies of the emitted photons (or another observable as commonly done in quantum mechanics [19][20][21][22][23][24]). Indeed, the eigenvalues equation involving the annihilation operator and the states of coherence is written below as:

$$\hat{\mathbf{a}}\,|\alpha\rangle = \alpha\,|\alpha\rangle = \sqrt{2}\sum_{-\infty}^{+\infty}\chi J_q(\chi)\,|\alpha\rangle \tag{25}$$

In Fig. 2 (Up panel) square of coherence Eq. 19 as well as $|\langle 0|\alpha\rangle|^2$ have been plotted as function of normalized Doppler-shift frequency. Interestingly in left-side up two peaks for $2 < \chi < 10$ The one of interest (blue line Q=2) because one finds minor peaks $\chi = 3, 6$ and $\chi = 9$ as well as one can see a large peak at $\chi = 10$ for Q=10. In the right-side it is easy to note that all lines are centered at $\chi = 0$ indicating that the classical view the system has null energy to emit photons at the Compton range, however one can see minor peaks for $\chi > 4$. In (Down panel) the square of amplitude $|\langle 0|\alpha\rangle|^2$ is plotted.Here one can see that the orange line Q=10, appears to be deeply degraded. In the contrary case. the blue line denoting Q=2 exhibits high values above 50%. Thus one can see that while the lowest values of Q exhibit high values, thus one can see that the orders of Bessel function dictated by:

$$x^2\frac{d^2 y}{dx^2} + x\frac{dy}{dx} + (x^2 - q^2)y = 0 \tag{26}$$

might be relevant for $n = 0$ yielding (after of dividing over x^2) one arrives to:

$$\frac{d^2 y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + y = 0 \tag{27}$$

that might be strongly related to the high values of $|\langle 0|\alpha\rangle|^2$

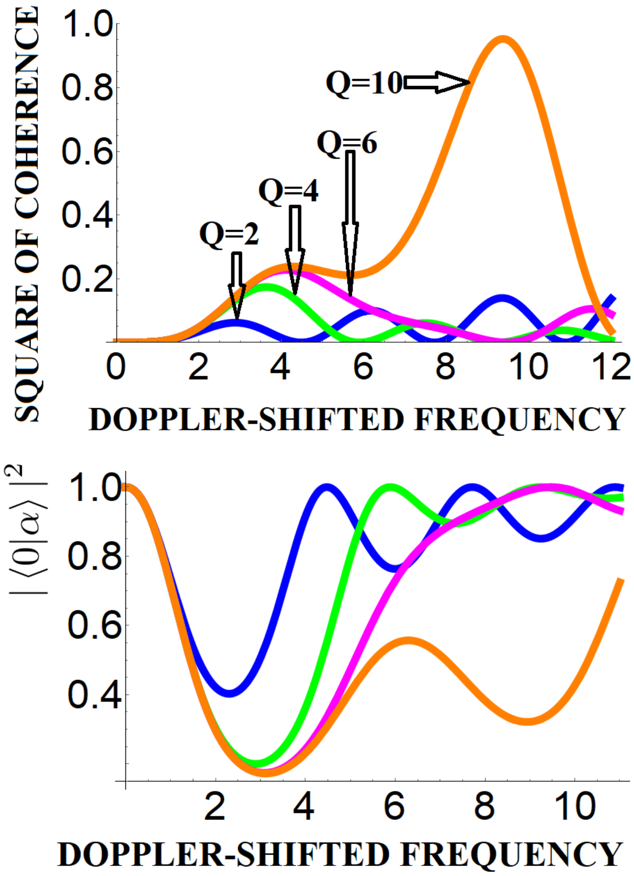

Fig. 2. (Up-Panel) Plotting of Eq. 19 Normalized to 1 for up to 4 Values of Integer Q. One can See that the Normalized Coherence Plotted as it Square Acquires a Similar form as the Radiation Intensities Plotted at Fig. 1. (Down-Panel) Plotting of Eq. 22 with Colors Same as Left-Panel Indicating that not any Peak at $\chi = 1$. Plots of Up and Down Panels were done with the Package [25].

A. *Identifying the Doppler-Shifted Frequency Orthogonal Basis*

The mathematical structure derived at Eq. 25 might be suggesting the existence of an orthogonal basis whose dependence would be given at the variable χ (see for example [26][27][28][29][30]). In fully accordance to the quantum mechanics formalism it is feasible to write down the completeness relationship for the normalized Doppler-shifted frequency χ as follows:

$$\int d\chi \, |\chi\rangle \langle\chi| = \mathbb{I}, \tag{28}$$

and in the other hand one has also that: $\int_{-\infty}^{\infty} d\alpha \, |\alpha\rangle \langle\alpha| = \mathbb{I}$ in conjunction to a discrete basis: $\sum_q |q\rangle \langle q| = \mathbb{I}$. In this way one can combine all these completeness relationships to arrive to:

$$\mathbb{I} \otimes \mathbb{I} = \sum_q \int d\chi \, \langle\chi|q\rangle \, |\chi\rangle \langle q| \, . \tag{29}$$

Now, one can include the coherent states also at the chain of multiplication of unitary operators in the form:

$$\begin{aligned} \mathbb{I} \otimes \mathbb{I} \otimes \mathbb{I} &= \int_{-\infty}^{\infty} d\alpha \, |\alpha\rangle \langle\alpha| \sum_q \int d\chi \, \langle\chi|q\rangle \, |\chi\rangle \langle q| \\ &= \sum_q \int d\chi \int_{-\infty}^{\infty} d\alpha \, |\alpha\rangle \langle\chi|q\rangle \langle\alpha|\chi\rangle \langle q| \end{aligned}$$

and the multiplication by the ket $|q\rangle$ in both sides one arrives to:

$$\begin{aligned} \mathbb{I} \otimes \mathbb{I} \otimes \mathbb{I} \, |q\rangle &= \sum_q \int d\chi \int_{-\infty}^{\infty} d\alpha \, |\alpha\rangle \langle\chi|q\rangle \langle\alpha|\chi\rangle \langle q|q\rangle \\ &= \sum_q \int d\alpha \, |\alpha\rangle \langle\alpha|\chi\rangle \int_{-\infty}^{\infty} d\chi \, \langle\chi|q\rangle \end{aligned}$$

where $\langle q|q\rangle = 1$. Indeed with the assumption:

$$\langle\alpha|\chi\rangle = \left(\frac{\alpha}{\chi}\right) \tag{30}$$

$$\langle\chi|q\rangle = \left(\chi \frac{d(\chi J_q(\chi))}{d\chi}\right) \tag{31}$$

then one gets from Eq. 27 in a straightforward manner the integration over χ:

$$\int_{-\infty}^{\infty} d\chi \frac{d(\chi J_q(\chi))}{d\chi} = \chi J_q(\chi) \tag{32}$$

so that the integral over α is trivial:

$$\int_0^{\sqrt{2\sqrt{2}}} d\alpha\alpha = \sqrt{2} \tag{33}$$

and by putting these integrations into Eq.27 then one can see that Eq. 25 is restored:

$$\mathbb{I} \otimes \mathbb{I} \otimes \mathbb{I} \, |q\rangle = |q\rangle = \sqrt{2} \sum_{q=-\infty}^{+\infty} \chi J_q(\chi) \, |\alpha\rangle \tag{34}$$

and multiplying Eq. 34 by the $\langle q|$ in both sides and with the definition of the polynomial $\langle q|\alpha\rangle \Rightarrow \langle\alpha|q\rangle = \left(\alpha \frac{d(\alpha J_q(\alpha))}{d\alpha}\right)$ derived from Eq. 31, then one can arrive to:

$$\sqrt{2} \sum_{q=-\infty}^{+\infty} \chi J_q(\chi) \left(\alpha \frac{d(\alpha J_q(\alpha))}{d\alpha}\right) = \mathbb{I} \tag{35}$$

Therefore, the derivative can be carry out:

$$\sum_{q=-\infty}^{+\infty} \left[\alpha\chi J_q^2(\alpha) + \chi\alpha^2 J_q(\chi) \frac{dJ_q(\alpha)}{d\alpha}\right] = \frac{1}{\sqrt{2}} \tag{36}$$

IV. Conclusion

In this paper, the quantum coherence has been extracted from the backscattered radiation intensity obtained in classical electrodynamics. While the HK model has been used, the results of this paper confirms that in the super intense regime the classical picture can restore quantum effects in particular the coherence of emitted radiation that to some extent is encompassing with requirements of advanced experiments that require backscattered radiation at the GeV energies to create unseen states of matter. Because these results can be understood as preliminary, in a next work the derivations from the HK model and its validation with current theories of quantum optics [31][32] shall be done.

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